1. Consider a sequence $\{a_n;\ n \geq 1\}$ of real numbers, where

$$a_{n+1} = \frac{3}{2}a_n - \frac{1}{2}a_{n-1} \quad \text{for all} \quad n > 1.$$

   (a) Show that the sequence converges.

   (b) Also find the limiting value of the sequence in terms of $a_1$ and $a_2$. [8+4]

2. Let $f$ be a real valued function defined on $[0, \infty)$ such that $f$ is continuous on $[0, \infty)$, $f(0) = 0$ and $f'$ is non-decreasing on $(0, \infty)$. Define $g(x) = f(x)/x$ for all $x \in (0, \infty)$. Show that $g$ is non-decreasing on $(0, \infty)$. [12]

3. Let $\mathbf{A}_{m\times m} = \begin{pmatrix} \frac{1}{\sqrt{m}} & \frac{1}{\sqrt{m}} & \cdots\cdots & \frac{1}{\sqrt{m}} \\ & \mathbf{P}_{m-1\times m} & & \end{pmatrix}$ be an orthogonal matrix and $\mathbf{B}$ be an $m \times m$ symmetric matrix with rank $m-1$ and $\mathbf{B}\mathbf{1}_m = \mathbf{0}$, where $\mathbf{1}_m = (1, 1, \ldots, 1)^T$ denotes the $m$-dimensional vector with all elements equal to 1. Show that

   (a) $\mathbf{P}^T\mathbf{P} = \mathbf{I} - \frac{1}{m}\mathbf{1}_m\mathbf{1}_m^T$, where $\mathbf{I}$ is the $m \times m$ identity matrix,

   (b) rank of $\mathbf{PBP}^T$ is $m-1$.

   Note: For a matrix $\mathbf{M}$, its transpose is denoted by $\mathbf{M}^T$. [4+8]

4. A fair coin is tossed repeatedly and let $\mathcal{T}$ be the number of tosses till two consecutive tails are observed for the first time.

   (a) Show that

$$E(\mathcal{T} \mid \text{tail is observed in the first toss}) = 2 + \frac{1}{2}E(\mathcal{T}).$$

   (b) Find a similar formula for

$$E(\mathcal{T} \mid \text{head is observed in the first toss}).$$

   (c) Compute $E(\mathcal{T})$. [6+3+3]

5. Consider a population consisting of $k$ classes with proportions $p_1, p_2, \ldots, p_k$, where $p_i \in (0, 1)$ for every $i = 1, 2, \ldots, k$ and $p_1 + p_2 + \cdots + p_k = 1$. Let $N$ denote the number of classes not represented in a random sample of size $n$ drawn with replacement from the population. Find $E(N^2)$. [12]

6. Let $U$ and $V$ be two dependent discrete random variables, each being uniformly distributed on $\{1, 2, \ldots, k\}$. Let $W$ be another random variable having the same uniform distribution but independent of $U$ and $V$. Define a random variable $X = (V + W) \mod k$. Show that

   (a) $X$ is uniformly distributed on $\{0, 1, 2, \ldots, k-1\}$,

   (b) $U$ and $X$ are independent. [6+6]

7. Consider a data set $(x_1, y_1), (x_2, , y_2), \ldots, (x_{100}, y_{100})$, where $x_i = a$ for all $i \leq 50$ and $x_i = b$ for all $i > 50$ $(a \neq b)$. Two regression functions

$$y = \alpha_0 + \alpha_1 x \quad \text{and} \quad y = \beta_0 + \beta_1 x^3$$

   are fitted to this data set using the method of least squares. Which of these two models will lead to smaller residual sum of squares? Justify your answer. [12]

8. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed normal random variables with mean $\theta$ and variance 1, where $\theta \geq 0$. Find

   (a) the maximum likelihood estimator $\hat{\theta}_n$ of $\theta$,

   (b) the asymptotic distribution of $T_n = \sqrt{n}\hat{\theta}_n$ when $\theta = 0$. [4+8]

9. Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution. For each $j = 1, 2, \ldots, n$, let

$$R_j = \#\{i : \ X_i \leq X_j, \ 1 \leq i \leq n\}.$$

   Thus, $R_j$ is the number of random variables $X_i$ $(i = 1, 2, \ldots, n)$ which are less than or equal to $X_j$.

   (a) Find the correlation coefficient between $R_1$ and $R_n$.

   (b) For any fixed $k$ $(1 < k < n)$, find the correlation coefficient between $Y_1 = \sum_{j=1}^{k} R_j$ and $Y_2 = \sum_{j=k+1}^{n} R_j$. [9+3]

10. For $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, define $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_d^2}$.

(a) Show that $f(\mathbf{x}) = \|\mathbf{x}\|$ is a convex function.

(b) Let $\mathbf{X}$ be a $d$-dimensional random vector symmetrically distributed about the origin (i.e. $\mathbf{X}$ and $-\mathbf{X}$ have the same distribution). Show that $\psi(\boldsymbol{\theta}) = E\|\mathbf{X} - \boldsymbol{\theta}\|$ is minimized at $\boldsymbol{\theta} = \mathbf{0}$. [4+8]